# CRANBROOK

# MATHEMATICS EXTENSION 1

# 2006

## HIGHER SCHOOL CERTIFICATE

## TRIAL EXAMINATION

## General Instructions

- Reading time – 5 minutes
- Writing time – 2 hours
- All seven questions should be attempted
- Total marks available - 84
- All questions are worth 12 marks
- An approved calculator may be used
- All relevant working should be shown for each question
- Complete your work in 7 separate 4 page booklets

**Question 1** (12 marks) **Marks**

(a) Evaluate $\lim_{x \to 0} \frac{2\sin 2x}{x}$. **2**

(b) The polynomial $P(x) = 2x^3 + ax^2 + x + 2$ has a factor $(2x+1)$. Find the value of $a$. **2**

(c) Solve the inequality $\frac{1}{x-2} \geq 2$. **2**

(d) Using the substitution $u = x + 2$, find $\int \frac{x}{3}\sqrt{x+2}\, dx$. **2**

(e) Divide the interval $AB$ externally in the ratio $4:3$, where $A$ is the point $(2,-1)$ and $B$ is $(1,-3)$. **2**

(f) Find the obtuse angle between the lines $3x - y + 5 = 0$ and $2x + 3y - 1 = 0$. Give your answer correct to the nearest degree. **2**

**Question 2** (12 marks) **Marks**

(a) Differentiate $\cos^{-1}(\sin x)$. 2

(b) Find:

(i) $\int \frac{2}{\sqrt{1-9x^2}}\,dx$ 2

(ii) $\int \sin^2 \frac{x}{2}\,dx$. 2

(c) If $\log_{16} 2 = \log_x 15$, find $x$. 2

(d) Consider the function $f(x) = \frac{1}{2}\cos^{-1}(1-3x)$.

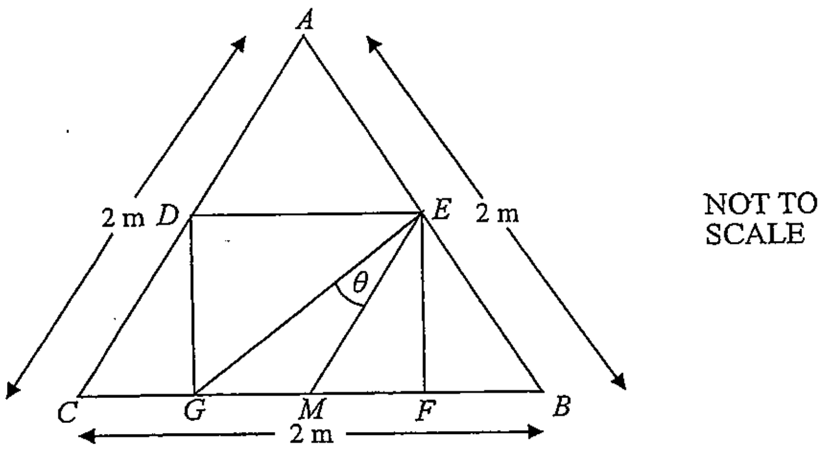
(i) State the domain and range of $f(x)$. 2

(ii) Hence sketch the graph of $y = f(x)$. 2

**Question 3** (12 marks) **Marks**

(a) Use the definition $f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$ to find $f'(x)$ where $f(x)=x^2-2x$. 2

(b)




The equilateral triangle $ABC$ has side lengths of 2 metres. The points $D$ and $E$ are the mid-points of $AC$ and $AB$ respectively and $GF$ lies on $CB$. Also, $DEFG$ is a rectangle. Point $M$ is the midpoint of $GF$. $\angle GEM=\theta$.

(i) Show that $GE=\frac{\sqrt{7}}{2}$ metres. 1

(ii) Show that $\cos\theta=\frac{5\sqrt{7}}{14}$. 2

(iii) Hence show that the area of $\Delta GEM$ is $\frac{\sqrt{3}}{8}$ square metres. 1

*Question 3 continues on the next page.*

**Question 3** (cont'd) **Marks**

(c) (i) Show that $f(x) = e^x - x^3 + 1$ has a zero between 4.4 and 4.6. **1**

(ii) Find an approximation; correct to 1 decimal place, for this zero using the method of halving the interval twice. **2**

(d) (i) Express $\sqrt{3}\cos x - \sin x$ in the form $R\cos(x+\alpha)$ where $0 < \alpha < \frac{\pi}{2}$ and $R > 0$. **2**

(ii) Hence, solve $\sqrt{3}\cos x - \sin x = 1$ for $0 \le x \le \frac{\pi}{2}$. **1**

**Question 4** (12 marks) **Marks**

(a) Prove $\tan^{-1}\frac{2}{3}+\cos^{-1}\frac{2}{\sqrt{5}}=\tan^{-1}\frac{7}{4}$. 3

(b) Find $\int\frac{\cos x\sin x}{\sqrt{2-\sin^2 x}}dx$ using the substitution $u=\sin x$. 3

(c) Sketch the curve $y=x+\frac{4}{x}$ showing clearly all the stationary points and asymptotes. Hence find the values of $k$ such that $x+\frac{4}{x}=k$ where there are no real roots. 3

(d) Use the method of mathematical induction to prove that $(1+1)+(2+3)+(3+5)+\ldots+(n+(2n-1))=\frac{1}{2}n(3n+1)$ where $n$ is a positive integer. 3

**Question 5** (12 marks) **Marks**

(a) A particle with displacement $x$ and velocity $v$, is moving in simple harmonic motion such that $\ddot{x} = -12x$. Initially, it is stationary at the point where $x = -4$.

(i) Show that **2**

$$v^2 = 12\left(16 - x^2\right).$$

(ii) By assuming the general form for $x$ or otherwise, find $x$ as a function of $t$. **2**

(b) The daily growth rate of a population, $P$, of a species of mosquito is proportional to the excess of the population over 5000.

i.e. $\dfrac{dP}{dt} = k(P - 5000)$.

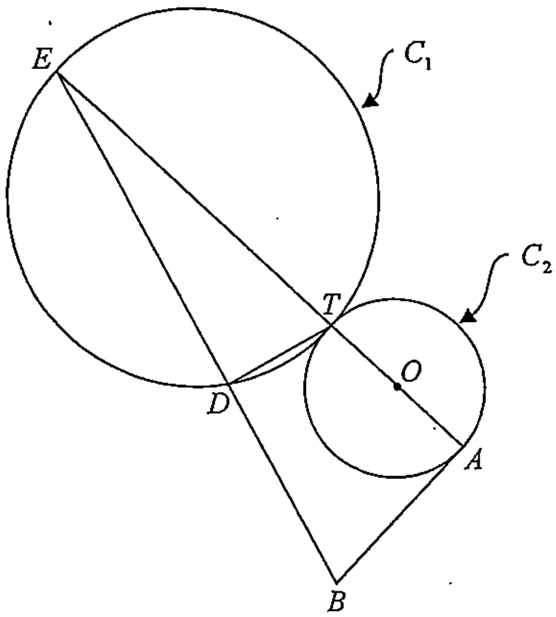
If $P = 5000 + Ae^{kt}$ satisfies this differential equation find:

(i) The values of $A$ and $k$ (in exact form) if initially the population was 5002 and after 6 days it was 25000. **2**

(ii) The mosquito population after 10 days to the nearest whole mosquito. **2**

**Question 5** (cont'd) **Marks**

(c)




NOT TO SCALE

Two circles $C_1$ and $C_2$ touch at $T$. The line $AE$ passes through $O$; the centre of $C_2$, and through $T$. The point $A$ lies on $C_2$ and $E$ lies on $C_1$. The line $AB$ is a tangent to $C_2$ at $A$, $D$ lies on $C_1$ and $BE$ passes through $D$. The radius of $C_1$ is $R$ and the radius of $C_2$ is $r$.

(i) Find the size of $\angle EDT$, giving reasons for your answer. **1**

(ii) If $DE = 2r$ find an expression for the length of $EB$ in terms of $r$ and $R$. **3**

**Question 6** (12 marks) **Marks**

(a) Prove that $\frac{d}{dx}(\operatorname{cosec} x) = -\operatorname{cosec} x \cot x$ 2

(b) A particle moves in a straight line so that its velocity $v$ at a position $x$ is given by $v = 9 + 4x^2$. It starts from a fixed point where $x = 0$ at a velocity of $9\,\text{m s}^{-1}$.

(i) Find its acceleration, $\ddot{x}$ as a function of the displacement, $x$. 1

(ii) Express its displacement, $x$, as a function of time, $t$. 3

(iii) Find the displacement after $\frac{\pi}{24}$ seconds from its starting point. 1

(c) (i) Find the equation of the normal at $P(at^2, 2at)$ on the parabola $y^2 = 4ax$. 2

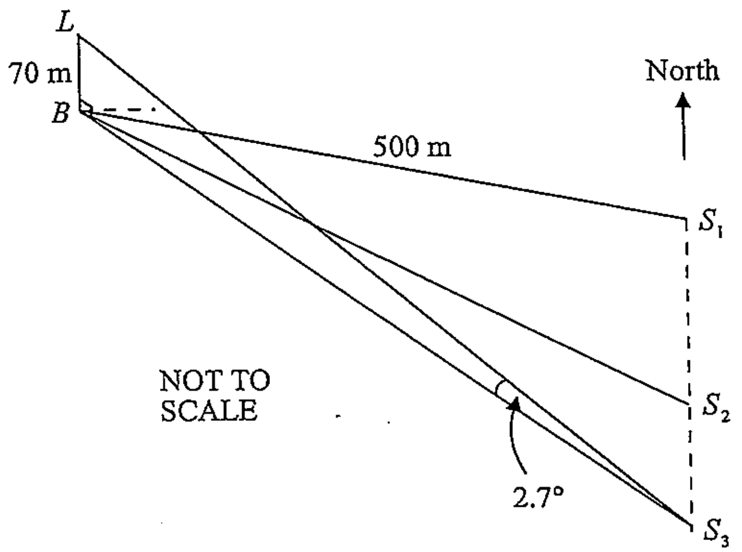
(ii) The normal intersects with the $x$-axis at point $Q$. Find the coordinates of $Q$ and hence find the coordinates of $R$ where $R$ is the midpoint of $PQ$. 2

(iii) Hence find the Cartesian equation of the locus of $R$. 1

**Question 7** (12 marks) **Marks**

(a) A life saver sits at point $L$, 70 m above sea level, in a lookout on a cliff. He spots a surfer lying on his board at point $S_1$; on a bearing of $120°$ in relation to the life savers position, and a distance of 500 m from point $B$ which is located at sea level directly below the life saver.

Twenty minutes later the life saver notes that the surfer is still lying on his surfboard but is now located at point $S_2$ on a bearing of $160°$ in relation to the life savers position and has moved in a line due south.




The life saver raises the alarm and twenty-three minutes after his initial sighting a rescue boat is launched from point $B$. From point $S_3$, which is the point at which the rescue boat reaches the surfer, the angle of elevation of the life saver in the tower is $2.7°$.

Assume that the rescue boat has travelled at a constant speed and in a straight line to reach the surfer and that the surfer has drifted at a constant speed in a direction due south since the initial sighting.

(i) Show that the speed at which the surfer was drifting south was 2.82 km/h (correct to 2 decimal places). **2**

(ii) Find the speed at which the rescue boat was moving. Express your answer in km/h correct to 2 decimal places. **3**

**Question 7** (cont'd) Marks

(b)

In the air polluted city of Thaams, a resident asthma sufferer is told by his doctor that he can only go outside his home when the pollution reading is less than 60 parts per million. For each day in Thaams the pollution is lowest at 5am with a reading of 10 parts per million and highest at 2pm with a reading of 90 parts per million.

(i) Assuming that the level of pollution readings is simple harmonic and of the form $\ddot{x} = -n^2(x-b)$, where $x = b$ is the centre of motion and $x = a$ is the amplitude, show that $x = b - a\cos nt$ satisfies this equation for simple harmonic motion. **2**

(ii) Hence determine the earliest time interval after 5am in the day that the asthma sufferer is **not** to go outside his home (to nearest minute). **5**

## STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x, \quad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

NOTE : $\ln x = \log_e x, \quad x > 0$

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**SOLUTIONS**
**2006**

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**Question 1** (12 marks)

(a) $$\lim_{x\to 0}\frac{2\sin 2x}{x} = 4\lim_{x\to 0}\frac{\sin 2x}{2x}$$
$$= 4\times 1 \text{ since } \lim_{x\to 0}\frac{\sin x}{x} = 1$$
$$= 4$$

$\lim_{x\to 0}\frac{2\sin 2x}{2x}\times 2$

| 2 marks | Correct answer |
|---|---|
| 1 mark | Obtaining $4\times\lim_{x\to 0}\frac{\sin 2x}{2x}$ |

(b) $P(x) = 2x^3 + ax^2 + x + 2$

Now, $P\left(-\frac{1}{2}\right) = 0$

so $P\left(-\frac{1}{2}\right) = 2\left(-\frac{1}{2}\right)^3 + a\left(-\frac{1}{2}\right)^2 + \left(-\frac{1}{2}\right) + 2 = 0$

$2\left(-\frac{1}{8}\right) + a\left(\frac{1}{4}\right) - \frac{1}{2} - 2 =$

$$-\frac{1}{4} + \frac{1}{4}a - \frac{1}{2} + 2 = 0$$
$$\frac{1}{4}a = -\frac{5}{4}$$
$$a = -5$$

| 2 marks | Correct answer |
|---|---|
| 1 mark | Obtaining correct expression for $P\left(-\frac{1}{2}\right)$ and then equating it to zero |

(c)

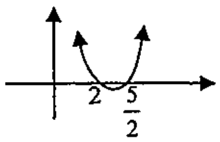
$$\frac{1}{x-2} \ge 2, \quad x \ne 2$$

$$x-2 \ge 2(x-2)^2$$

$$2(x-2)^2 - (x-2) \le 0$$

$$(x-2)[2(x-2)-1] \le 0$$

$$(x-2)(2x-5) \le 0$$




so $2 < x \le \frac{5}{2}$ since $x \ne 2$

$(x-2) - 2(x-2)^2 \ge 0$
$(x-2)[1-2(x-2)] \ge 0$
$(x-2)(1-2x+4) \ge 0$
$(x-2)(5-2x) \ge 0$
$x = 2 \quad x = \frac{5}{2}$

$2 < x \le \frac{5}{2}$

| | |
|---|---|
| 2 marks | Correct answer |
| 1 mark | Obtaining $(x-2)[2(x-2)-1] \le 0$ |

(d)

$$\int \frac{x}{3}\sqrt{x+2}\,dx = \frac{1}{3}\int (u-2)u^{\frac{1}{2}}\,du \qquad \text{where } u = x+2,\ \frac{du}{dx} = 1 \text{ and } x = u-2$$

$$= \frac{1}{3}\int (u^{\frac{3}{2}} - 2u^{\frac{1}{2}})du \ \checkmark$$

$$= \frac{1}{3}\left(\frac{2}{5}u^{\frac{5}{2}} - \frac{4}{3}u^{\frac{3}{2}}\right)$$

$$= \frac{2}{3}\left\{\frac{1}{5}(x+2)^{\frac{5}{2}} - \frac{2}{3}(x+2)^{\frac{3}{2}}\right\} \ \checkmark$$

| | |
|---|---|
| 2 marks | Correct answer |
| 1 mark | Obtaining $\frac{1}{3}\int (u^{\frac{3}{2}} - 2u^{\frac{1}{2}})du$ |

(e)

$A(2,-1),\ B(1,-3)$

$4:-3$

$$\left(\frac{4(1)-3(2)}{4-3}, \frac{4(-3)-3(-1)}{4-3}\right)$$

$$= (-2,-9)$$

| | |
|---|---|
| 2 marks | Correct answer |
| 1 mark | One correct coordinate |

(f) $3x-y+5=0,\ 2x+3y-1=0$

$$m_1=3,\quad m_2=-\frac{2}{3}$$

$$\tan\theta=\frac{3+\frac{2}{3}}{1+3\left(-\frac{2}{3}\right)} \quad \text{OR} \quad \tan\theta=\left|\frac{3+\frac{2}{3}}{1+3\left(-\frac{2}{3}\right)}\right|$$

$$=-3\frac{2}{3} \qquad\qquad \theta=75^\circ$$

$\theta=105^\circ$ so the obtuse angle is $105^\circ$

| | |
|---|---|
| 2 marks | Correct answer |
| 1 mark | Obtaining $m_1=3$ and $m_2=-\frac{2}{3}$ |

**Question 2** (12 marks)

(a) $$\frac{d}{dx}\left(\cos^{-1}(\sin x)\right)=\frac{-\cos x}{\sqrt{1-\sin^2 x}}$$
$$=\frac{-\cos x}{|\cos x|}$$
$$=\begin{cases}-1, & \text{if } \cos x>0\\ 1, & \text{if } \cos x<0\end{cases}$$

| 2 marks | Correct answer |
|---|---|
| 1 mark | Obtaining $\frac{d}{dx}\left(\cos^{-1}(\sin x)\right)=\frac{-\cos x}{\sqrt{1-\sin^2 x}}$ |

(b) (i) $$\int\frac{2dx}{\sqrt{1-9x^2}}=\frac{2}{3}\int\frac{3dx}{\sqrt{1-(3x)^2}}$$
$$=\frac{2}{3}\sin^{-1}(3x)+c$$

| 2 marks | Correct answer |
|---|---|
| 1 mark | Obtaining $\frac{2}{3}\int\frac{3dx}{\sqrt{1-(3x)^2}}$ |

(ii) $$\int\sin^2\frac{x}{2}\,dx=\int\left(\frac{1}{2}-\frac{1}{2}\cos x\right)dx$$
$$=\frac{1}{2}x-\frac{1}{2}\sin x+c$$

| 2 marks | Correct answer |
|---|---|
| 1 mark | Obtaining $\int\left(\frac{1}{2}-\frac{1}{2}\cos x\right)dx$ |

(c) $\log_{16}2=\log_x 15$ $\quad\therefore \log_{16}16^{\frac{1}{4}}=\log_x 15$ $\quad\therefore \frac{1}{4}=\log_x 15$

$\therefore x^{\frac{1}{4}}=15$ $\quad\therefore x=15^4=50625$

| 2 marks | Correct answer |
|---|---|
| 1 mark | Giving $\frac{1}{4}=\log_x 15$ |

(d) $f(x)=\frac{1}{2}\cos^{-1}(1-3x)$

(i) Domain:

$-1 \le 1-3x \le 1$

$-2 \le -3x \le 0$

$\frac{2}{3} \ge x \ge 0$

so $0 \le x \le \frac{2}{3}$

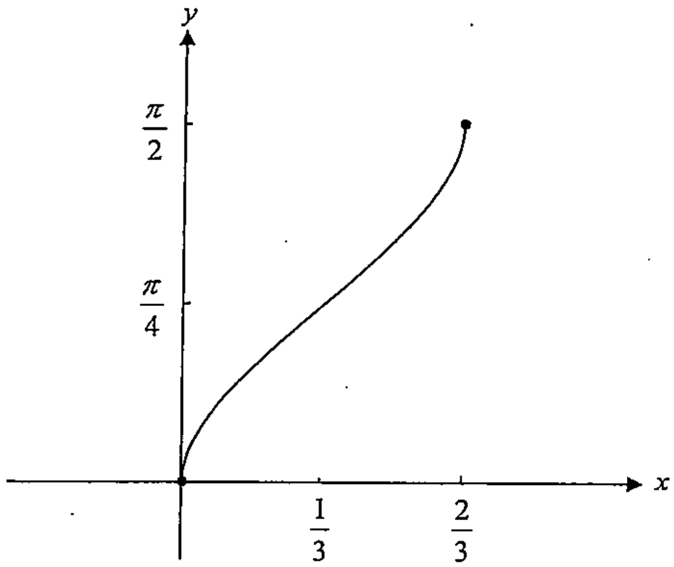
Range:

$0 \le 2y \le \pi$

$0 \le y \le \frac{\pi}{2}$

| | |
|---|---|
| 2 marks | Correct answer |
| 1 mark | Correct domain or correct range |

(ii)




Test $x=0$, $f(0)=\frac{1}{2}\cos^{-1}1$

$=0$

$x=\frac{2}{3}$, $f\left(\frac{2}{3}\right)=\frac{1}{2}\cos^{-1}(-1)$

$=\frac{\pi}{2}$

| | |
|---|---|
| 2 marks | Correct graph |
| 1 mark | Correct shape without endpoints marked or point of inflection labelled |

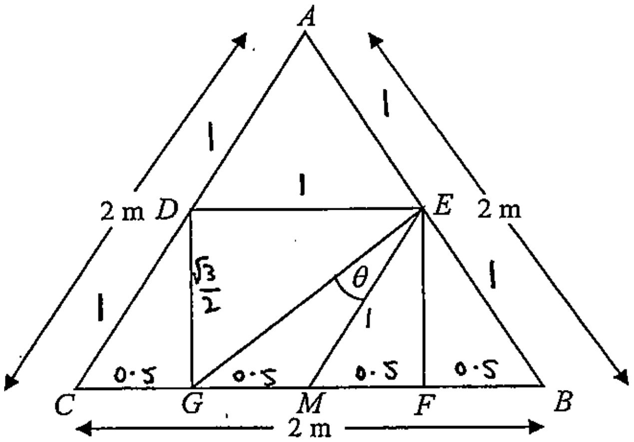
**Question 3** (12 marks)

(a)

$$f(x) = x^2 - 2x$$
$$f(x+h) = (x+h)^2 - 2(x+h)$$
$$= x^2 + 2hx + h^2 - 2x - 2h$$
$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$
$$= \lim_{h \to 0} \frac{x^2 + 2hx + h^2 - 2x - 2h - x^2 + 2x}{h}$$
$$= \lim_{h \to 0} \frac{2hx + h^2 - 2h}{h}$$
$$= \lim_{h \to 0} (2x + h - 2)$$
$$= 2x - 2$$

| | |
|---|---|
| 2 marks | Correctly derived derivative |
| 1 mark | Correctly substituting into $f'(x)$ |

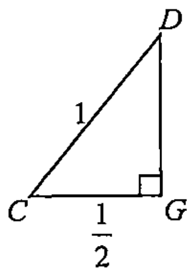
(b) (i)




NOT TO SCALE

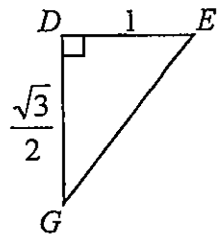
In $\Delta CDG$

$$DG = \sqrt{1 - \left(\frac{1}{2}\right)^2}$$
$$= \frac{\sqrt{3}}{2} \text{ m}$$




In $\Delta DEG$

$$GE = \sqrt{1^2 + \left(\frac{\sqrt{3}}{2}\right)^2}$$
$$= \frac{\sqrt{7}}{2} \text{ metres}$$

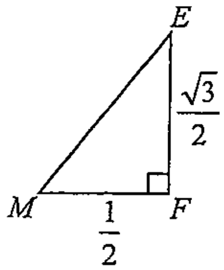



as required

| | |
|---|---|
| 1 mark | Correct answer |

(ii) In $\Delta EFM$

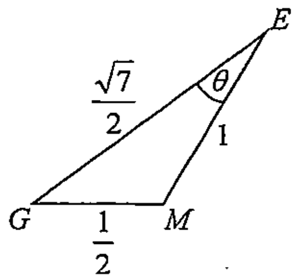
$$EM = \sqrt{\frac{1}{4}+\frac{3}{4}}$$
$$= 1$$




In $\Delta GEM$,

$$\left(\frac{1}{2}\right)^2 = \frac{7}{4}+1-2\times\frac{\sqrt{7}}{2}\times 1\cos\theta$$
$$-\frac{5}{2} = -\sqrt{7}\cos\theta$$
$$\cos\theta = \frac{5\sqrt{7}}{14} \text{ as required}$$

$$\cos\theta = \frac{\left(\frac{\sqrt{7}}{2}\right)^2+1^2-\left(\frac{1}{2}\right)^2}{2\times\frac{\sqrt{7}}{2}\times 1} = \frac{\frac{7}{4}+1-\frac{1}{4}}{\sqrt{7}} = \frac{\frac{5}{2}}{\sqrt{7}}\times\frac{\sqrt{7}}{\sqrt{7}} = \frac{5\sqrt{7}}{14}$$




| | |
|---|---|
| 2 marks | Correct answer |
| 1 mark | Finding *EM* and attempting to use the cosine rule |

(iii)

$$\text{Area} = \frac{1}{2}\times GE\times EM\sin\theta$$
$$= \frac{1}{2}\times\frac{\sqrt{7}}{2}\times 1\sin\theta$$
$$= \frac{\sqrt{7}\sqrt{21}}{56}$$
$$= \frac{7\sqrt{3}}{56}$$
$$= \frac{\sqrt{3}}{8}\text{m}^2 \text{ as required.}$$

$$\sin^2\theta+\cos^2\theta = 1$$
$$\sin\theta = \sqrt{1-\frac{175}{196}}$$
$$= \frac{\sqrt{21}}{14}$$

| | |
|---|---|
| 1 mark | Correct answer |

(c) (i)

$$f(x) = e^x - x^3 + 1$$
$$f(4\cdot 4) = -2.7331\ldots$$
$$f(4\cdot 6) = 3\cdot 1483\ldots$$

The sign of the function changes between $x = 4\cdot 4$ and $x = 4\cdot 6$ and hence the zero must lie between $x = 4\cdot 4$ and $4\cdot 6$.

| | |
|---|---|
| 1 mark | Correct answer |

(ii) By the method of halving the interval $x_1 = \dfrac{4.4+4.6}{2} = 4.5$

Now $f(4.5) = -0.10786\ldots < 0$

$\therefore$ As $f(4.5)$ and $f(4.6)$ have opposite signs then the zero is in the interval $4.5 < x < 4.6$.

Now $x_2 = \dfrac{4.5+4.6}{2} = 4.55 = 4.6(1\,\text{d p})$

$\therefore$ The zero is 4.6 to 1 decimal place by using the halving the interval method twice.

| | |
|---|---|
| 2 marks | Correct answer |
| 1 mark | Using the correct method but with an arithmetic mistake |

(d) (i) $R\cos(x+\alpha) = R\cos x\cos\alpha - R\sin x\sin\alpha$ for all $x$

Now, $\sqrt{3}\cos x - \sin x = R\cos x\cos\alpha - R\sin x\sin\alpha$

so, $\sqrt{3} = R\cos\alpha$

and $1 = R\sin\alpha$

So, $\tan\alpha = \dfrac{1}{\sqrt{3}}$

$\alpha = \dfrac{\pi}{6}$

Also $R^2\cos^2\alpha + R^2\sin^2\alpha = 4$

$R = 2, R > 0$

So $\sqrt{3}\cos x - \sin x = 2\cos\left(x + \dfrac{\pi}{6}\right)$

$R = \sqrt{3+1} = 2$

$\tan\alpha = \dfrac{1}{\sqrt{3}}$

$\alpha = \dfrac{\pi}{6}$

$2\cos\left(x+\dfrac{\pi}{6}\right)$

| | |
|---|---|
| 2 marks | Correct answer |
| 1 mark | Correctly finding $R$ or $\alpha$ |

(ii) $\sqrt{3}\cos x - \sin x = 1$

$2\cos\left(x+\dfrac{\pi}{6}\right) = 1$ from part (i)

$\cos\left(x+\dfrac{\pi}{6}\right) = \dfrac{1}{2}$

$x + \dfrac{\pi}{6} = \dfrac{\pi}{3}$

$x = \dfrac{\pi}{6}$

| | |
|---|---|
| 1 mark | Correct answer |

**Question 4** (12 marks)

(a) Prove $\tan^{-1}\frac{2}{3}+\cos^{-1}\frac{2}{\sqrt{5}}=\tan^{-1}\frac{7}{4}$

$$\tan\left(\tan^{-1}\frac{2}{3}+\cos^{-1}\frac{2}{\sqrt{5}}\right)=\frac{7}{4}$$

let $A=\tan^{-1}\frac{2}{3}$ and $B=\cos^{-1}\frac{2}{\sqrt{5}}$

so $\tan A=\frac{2}{3}$ and $\cos B=\frac{2}{\sqrt{5}}$

so $\tan B=\frac{1}{2}$

$$\begin{aligned} LHS &= \tan(A+B) \\ &= \frac{\tan A+\tan B}{1-\tan A\tan B} \\ &= \frac{\frac{2}{3}+\frac{1}{2}}{1-\frac{2}{3}\times\frac{1}{2}} \\ &= \frac{\frac{7}{6}}{\frac{2}{3}} \\ &= \frac{7}{4} \\ &= RHS \end{aligned}$$

| | |
|---|---|
| 3 marks | Correct answer |
| 2 marks | Obtaining $\tan A=\frac{2}{3}$ and $\tan B=\frac{1}{2}$ |
| 1 mark | Obtaining $\tan\left(\tan^{-1}\frac{2}{3}+\cos^{-1}\frac{2}{\sqrt{5}}\right)=\frac{7}{4}$ |

(b) $\int \frac{\cos x \sin x}{\sqrt{2-\sin^2 x}}\,dx$

let $u = \sin x$ so $\frac{du}{dx} = \cos x$

$dx = \frac{du}{\cos x}$

$= \int \frac{\cos x \times u}{\sqrt{2-u^2}} \frac{du}{\cos x}$

$= \int u\left(2-u^2\right)^{-\frac{1}{2}} du$

$= -\frac{1}{2}\int -2u\left(2-u^2\right)^{-\frac{1}{2}} du$

$= -\frac{1}{2}\frac{\left(2-u^2\right)^{\frac{1}{2}}}{\frac{1}{2}}$

$= -\sqrt{2-u^2}$

$= -\sqrt{2-\sin^2 x} + c$

| | |
|---|---|
| 3 marks | Correct answer |
| 2 marks | Obtaining $\int u\left(2-u^2\right)^{-\frac{1}{2}} du$ |
| 1 mark | Obtaining $dx = \frac{du}{\cos x}$ |

(c) $y = x + \frac{4}{x}$

$\frac{dy}{dx} = 1 - \frac{4}{x^2}$

$\frac{d^2y}{dx^2} = \frac{8}{x^3}$

Stationary points occur when $\frac{dy}{dx} = 0$.

When $1 - \frac{4}{x^2} = 0$

$x^2 = 4$

$x = \pm 2$

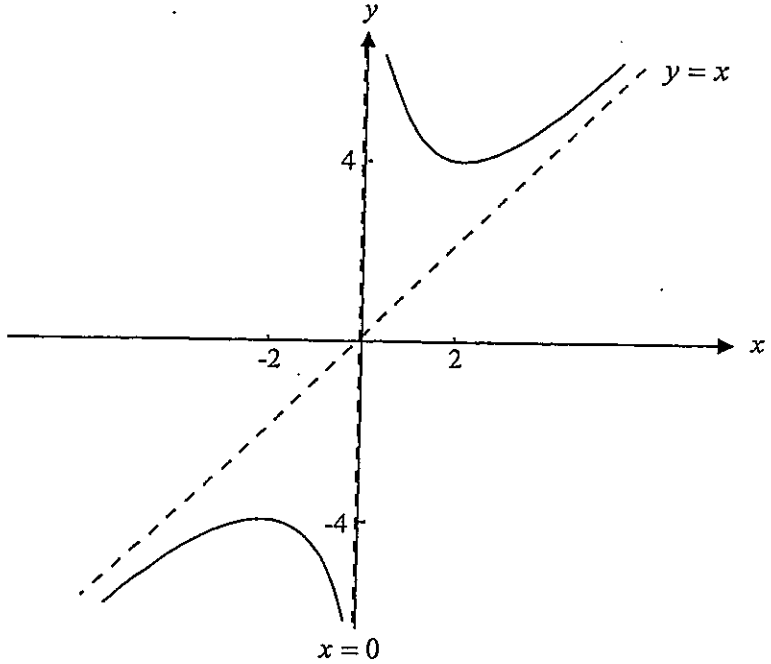
$x = 2,\ y = 2 + \frac{4}{2} = 4$

$x = -2,\ y = -2 + \frac{4}{-2} = -4$

At the point $(-2,-4)$, $\frac{d^2y}{dx^2} < 0$ so we have a max. at this point.

At the point $(2,4)$, $\frac{d^2y}{dx^2} > 0$ so we have a min. at this point.

Asymptotes $x = 0,\ y = x$




From the graph we see that there are no real roots for $x + \frac{4}{x} = k$ when $-4 < k < 4$.

| | |
|---|---|
| 3 marks | Correct answer and graph |
| 2 marks | Correct graph |
| 1 mark | Finding correct local minimum and local maximum OR correct asymptotes |

(d) $(1+1)+(2+3)+(3+5)+\ldots+(n+(2n-1))=\frac{1}{2}n(3n+1)$, $n$ is a positive integer

Step 1

For $n=1$,

$LHS = 1+1$
$= 2$

$RHS = \frac{1}{2}(1)(3+1)$
$= 2$
$\therefore LHS = RHS$

Hence true for $n=1$

Step 2: Assume true for $n=k$,

$(1+1)+(2+3)+(3+5)+\ldots+(k+(2k-1))=\frac{1}{2}k(3k+1)$ (*)

Prove true for $n=k+1$,

$LHS = (1+1)+(2+3)+\ldots+(k+(2k-1))+(k+1+(2(k+1)-1))$
$=\frac{1}{2}k(3k+1)+(k+1+2k+1)$ (from *)
$=\frac{3}{2}k^2+\frac{1}{2}k+3k+2$
$=\frac{3}{2}k^2+\frac{7}{2}k+2$
$=\frac{1}{2}(3k^2+7k+4)$
$=\frac{1}{2}(k+1)(3k+4)$
$=\frac{1}{2}(k+1)(3(k+1)+1)$

So it is true for $n=(k+1)$

Step 3 Since it is true for $n=1, n=2$ and so on and it is true for $n=k+1$ by assuming true for $n=k$, therefore by mathematical induction, it is true for all $n$ where $n$ is a positive integer.

| | |
|---|---|
| 3 marks | Correct proof |
| 2 marks | Verifies the case for $n=1$ and attempts to prove that it is true for $n=k+1$ |
| 1 mark | Verifies the case for $n=1$ |

**Question 5** (12 marks)

(a) (i)

$$\ddot{x} = -12x$$

$$\frac{d}{dx}\left(\frac{1}{2}v^2\right) = -12x$$

$$\frac{1}{2}v^2 = -12\int x\,dx$$

$$\frac{1}{2}v^2 = \frac{-12x^2}{2} + c$$

When $v = 0,\ x = -4$

$$0 = -96 + c$$

$$c = 96$$

So, $\frac{1}{2}v^2 = -6x^2 + 96$

$$v^2 = -12x^2 + 192$$

$$v^2 = 12\left(16 - x^2\right)$$

as required.

| 2 marks | Correct answer |
|---|---|
| 1 mark | Obtaining $\frac{1}{2}v^2 = -6x^2 + c$ OR<br>Correct substitution of initial conditions into incorrect function |

(ii) Since the motion starts at the 'negative' end of the path of motion we use the general form $x = -a\cos nt$.
Now, $\ddot{x} = -12x$ so $n^2 = 12$ and so $n = 2\sqrt{3},\ n > 0$.
Also, the particle is stationary at $x = -4$ and since $\ddot{x} = -12x$, the centre of motion is at $x = 0$ so $a = 4$.
So $x = -4\cos\left(2\sqrt{3}t\right)$.

| 2 marks | Correct answer |
|---|---|
| 1 mark | Finding $a$ and $n$ correctly OR giving correct general form for $x$ |

(b) (i) When $t = 0$ $P = 5002$

$\therefore 5002 = 5000 + Ae^0$

$\therefore A = 2$

When $t = 6$ $P = 25000$ $\therefore 25000 = 5000 + 2e^{6k}$

$\therefore k = \frac{1}{6}\ln 10\,000$

| 2 marks | Correct answers |
|---|---|
| 1 mark | Obtaining only $A = 2$ or $k = \frac{1}{6}\ln 10\,000$ |

(ii) Now $P = 5000 + 2e^{(\frac{1}{6}\ln 10000)t}$

When $t = 10$ $P = 5000 + 2e^{(\frac{1}{6}\ln 10000)10}$

$\therefore P = 5000 + 2e^{\ln 10000^{\frac{10}{6}}}$

$\therefore P = 5000 + 2(10\,000)^{\frac{10}{6}}$

$\therefore P = 9\,288\,178$ (to nearest whole mosquito)

| 2 marks | Correct answer |
|---|---|
| 1 mark | Showing $P = 5000 + 2e^{\ln 10000^{\frac{10}{6}}}$ |

(c) (i) Since $C_1$ and $C_2$ touch at $T$ and $AE$ passes through $O$; the centre of $C_2$, and through $T$, then $AE$ also passes through the centre of $C_1$. So $TE$ is a diameter of $C_1$. (When circles touch, the line through the centres passes through the point of contact.)

So $\angle EDT = 90^\circ$ (Angles in a semi-circle are right angles.)

| | |
|---|---|
| 1 mark | Correct answer |

(ii) $\angle EAB = 90^\circ$ (The tangent to a circle is perpendicular to the radius drawn to the point of contact.)

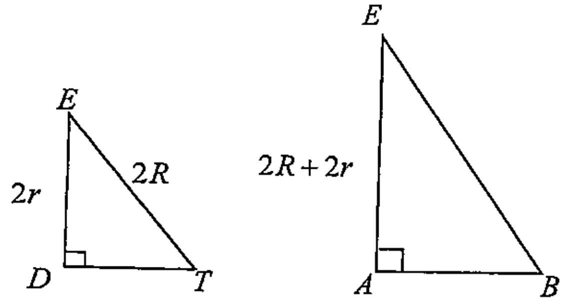
In $\Delta$'s $ABE$ and $DTE$,

$\angle EDT = \angle EAB = 90^\circ$ (from part (i) and from above)

$\angle DET$ is common

$\angle ETD = \angle EBA$ (Angles in a triangle add to give $180^\circ$.)

So $\Delta ABE$ is similar to $\Delta DTE$. (Three corresponding angles are equal.)




Now, $DE = 2r$, $ET = 2R$, and $EA = 2R + 2r$

Since the $\Delta$'s are similar,

$$\frac{EB}{ET} = \frac{EA}{ED}$$

$$EB = 2R\left(\frac{2R+2r}{2r}\right)$$

$$= \frac{2R(R+r)}{r}$$

| | |
|---|---|
| 3 marks | Correct answer including showing that $\Delta ABE$ is similar to $\Delta DTE$ |
| 2 marks | Showing that $\Delta ABE$ is similar to $\Delta DTE$ |
| 1 mark | Stating that $\angle EAB = 90^\circ$ |

**Question 6** (12 marks)

(a)
$$\begin{aligned}
LHS &= \frac{d}{dx}(\operatorname{cosec} x) \\
&= \frac{d}{dx}\left(\frac{1}{\sin x}\right) \\
&= \frac{d}{dx}\left([\sin x]^{-1}\right) \\
&= -[\sin x]^{-2}\cos x \\
&= -\frac{1}{\sin x}\frac{\cos x}{\sin x} \\
&= -\operatorname{cosec} x \cot x \\
&= RHS
\end{aligned}$$

| | |
|---|---|
| 2 marks | Correct answer |
| 1 mark | Writing LHS as $-[\sin x]^{-2}\cos x$ |

(b) (i)
$$\begin{aligned}
v &= 9+4x^2 \\
\ddot{x} &= \frac{d}{dx}\left(\frac{1}{2}v^2\right) \\
&= \frac{d}{dx}\left(\frac{1}{2}\left(9+4x^2\right)^2\right) \\
&= \left(9+4x^2\right)(8x) \\
&= 8x\left(9+4x^2\right)
\end{aligned}$$

| | |
|---|---|
| 1 mark | Correct answer |

(ii) $v = 9 + 4x^2$

$$\frac{dx}{dt} = 9 + 4x^2$$

$$\frac{dt}{dx} = \frac{1}{9 + 4x^2}$$

$$t = \int \frac{1}{9 + 4x^2}\, dx$$

$$= \frac{1}{4} \int \frac{1}{\frac{9}{4} + x^2}\, dx$$

$$= \frac{1}{4} \times \frac{2}{3} \tan^{-1}\left(\frac{2x}{3}\right) + c$$

$$= \frac{1}{6} \tan^{-1}\left(\frac{2x}{3}\right) + c$$

$t = 0$ and $x = 0$, so $c = 0$

$$t = \frac{1}{6} \tan^{-1}\left(\frac{2x}{3}\right)$$

$$6t = \tan^{-1}\left(\frac{2x}{3}\right)$$

$$\frac{2x}{3} = \tan 6t$$

$$x = \frac{3}{2} \tan 6t$$

| | |
|---|---|
| 3 marks | Correct answer |
| 2 marks | Obtaining $t = \frac{1}{6} \tan^{-1}\left(\frac{2x}{3}\right)$ |
| 1 mark | Obtaining $t = \int \frac{1}{9 + 4x^2}\, dx$ |

(iii) $t = \frac{\pi}{24}$, $x = \frac{3}{2} \tan 6\left(\frac{\pi}{24}\right)$

$$= \frac{3}{2} \tan \frac{\pi}{4}$$

$$= 1\frac{1}{2}\text{ m}$$

The displacement is 1.5m after $\frac{\pi}{24}$ seconds from its starting point.

| | |
|---|---|
| 1 mark | Correct answer |

(c) (i) $x = at^2, \quad y = 2at$

$$\frac{dx}{dt} = 2at \quad \frac{dy}{dt} = 2a$$

$$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$$

$$= 2a \times \frac{1}{2at}$$

$$= \frac{1}{t}$$

$m_T = \frac{1}{t}$ so $m_N = -t \quad (\text{since } m_1 m_2 = -1)$

Equation of normal at $P(at^2, 2at)$ is given by

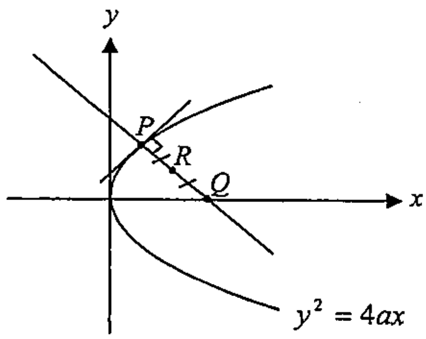
$$y - 2at = -t(x - at^2)$$

$$= -tx + at^3$$

$$tx + y = 2at + at^3$$

| | |
|---|---|
| 2 marks | Correct answer |
| 1 mark | Obtaining the correct gradient of the normal |

(ii)




At $Q$, $y = 0$.

sub $y = 0$ into $tx + y = 2at + at^3$

$$tx = 2at + at^3$$

$$x = a(2 + t^2)$$

So we have $Q(a(2+t^2), 0)$

and $R\left(\frac{a(2+t^2)+at^2}{2}, \frac{0+2at}{2}\right)$

$$R(a(1+t^2), at)$$

| | |
|---|---|
| 2 marks | Correct answer |
| 1 mark | Obtaining one correct coordinate of $R$ OR finding the correct coordinates of $Q$ |

(iii) At $R, x = a(1+t^2)$ and $y = at$

$$\text{so } x = a\left(1+\frac{y^2}{a^2}\right) \text{ and } t = \frac{y}{a}$$

$$\frac{x}{a} = 1+\frac{y^2}{a^2}$$

$$\frac{y^2}{a^2} = \frac{x}{a}-1$$

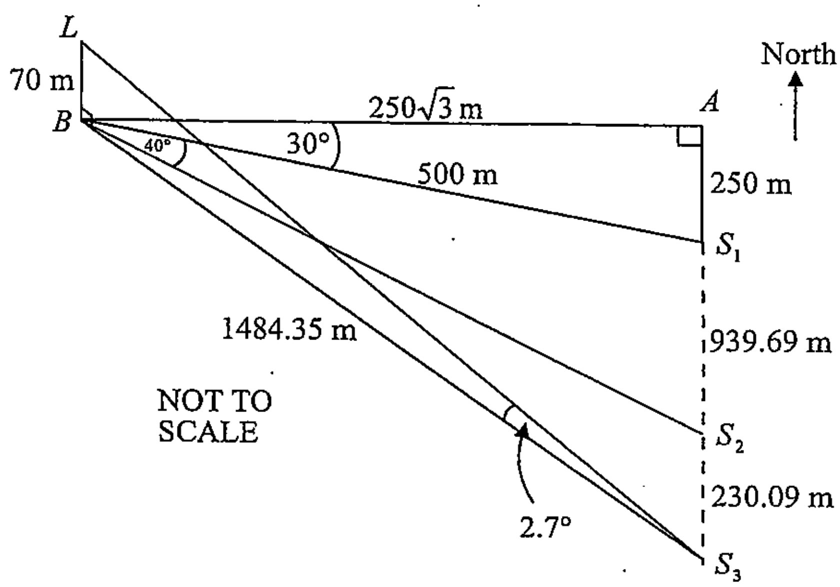
$$y^2 = ax - a^2$$

$$\text{so } y^2 = a(x-a)$$

| | |
|---|---|
| 1 mark | Correct answer |

**Question 7** (12 marks)

(a)




(i)

$$\text{In } \Delta ABS_1,\ AB = 500\cos 30^\circ$$
$$= 250\sqrt{3}$$
$$\text{Also, } \quad AS_1 = 500\sin 30^\circ$$
$$= 250$$
$$\text{In } \Delta ABS_2,\ AS_2 = 250\sqrt{3}\tan 70^\circ$$
$$= 1189.692...$$
$$\text{so, } \quad S_1S_2 = AS_2 - AS_1$$
$$= 939.692...$$
$$\text{Speed of surfer} = \frac{S_1S_2}{20\,\text{mins}}$$
$$= \frac{0.939692...}{0.\dot{3}}\text{ km/h}$$
$$= 2.82 \quad \text{km/h (correct to 2 decimal places)}$$

as required

| | |
|---|---|
| 2 marks | Correct answer |
| 1 mark | Making a reasonable attempt to find the distance $S_1S_2$ |

(ii) In $\Delta BLS_3$, $BS_3 = \dfrac{70}{\tan 2.7^\circ}$
$= 1484.346...$

In $\Delta ABS_3$, $AS_3 = \sqrt{(BS_3)^2 - (250\sqrt{3})^2}$
$= 1419.783...$

So $S_2S_3 = 1419.783 - 250 - 939.692 = 230.091$

The time taken for the surfer to drift from $S_2$ to $S_3$ is $\dfrac{0.230091}{2.82} = 0.08159\text{ h}$ (to 5 dec. places).

The rescue boat leaves 3 minutes or $\dfrac{3}{60} = 0.05$ hours after the lifesaver spotted the surfer at point $S_2$. Therefore the rescue boat took $0.08159 - 0.05 = 0.03159$ hours to cover the distance of 1.48435 km.

The speed of the rescue boat was therefore $\dfrac{1.48435}{0.03159} = 46.99$ km/h (correct to 2 decimal places).

| | |
|---|---|
| 3 marks | Correct answer |
| 2 marks | Finding correctly the time taken for the surfer to drift from $S_2$ to $S_3$ |
| 1 mark | Finding correctly the distance from $S_2$ to $S_3$ |

(b)

(i) $\ddot{x} = -n^2(x-b)$ ..........(1)

If $x = b - a\cos nt$............(2)

$\therefore \dot{x} = an\sin nt$

$\therefore \ddot{x} = an^2\cos nt$............(3)

Sub (2) and (3) into (1):

$$\begin{aligned}\text{LHS} &= \ddot{x} \\ &= an^2\cos nt \\ &= n^2(a\cos nt) \\ &= n^2(b-x) \qquad \text{(from (2))} \\ &= -n^2(x-b) \\ &= \text{RHS}\end{aligned}$$

| | |
|---|---|
| 2 mark | Correctly proving the result |
| 1 mark | Partly proving the result |

(ii) Using $x = b - a\cos nt$

$\therefore b = \dfrac{10+90}{2} = 50$ and $a = \dfrac{90-10}{2} = 40$

Period, $T = \dfrac{2\pi}{n}$ $\quad \therefore 18 = \dfrac{2\pi}{n}$ $\quad \therefore n = \frac{\pi}{9}$

$\therefore x = 50 - 40\cos\dfrac{\pi t}{9}$

If $x = 60$ $\quad 60 = 50 - 40\cos\dfrac{\pi t}{9}$

$\therefore -\dfrac{1}{4} = \cos\dfrac{\pi t}{9}$

$\therefore \dfrac{\pi t}{9} = \cos^{-1}(-\dfrac{1}{4})$

$\therefore \dfrac{\pi t}{9} = \pi - \cos^{-1}\dfrac{1}{4}$ *or* $\pi + \cos^{-1}\dfrac{1}{4}$ (for 2nd, 3rd quadrants)

$\therefore t = 5.22......or\ 12.776......$

$\therefore t = $ 5hr 13 min 25.95...s or 12hr 46 min 34.05..s

$\therefore$ the earliest time interval from 5am that the asthma sufferer should not go outside his home is from 10.13am to 5.47pm (to the nearest minute) so that the pollution reading is less than 60 parts per million.

| | |
|---|---|
| 5 marks | Correctly determining the earliest time interval |
| 4 marks | Making only one mistake with the earliest time interval |
| 3 marks | Obtaining the equation $-\frac{1}{4}=\cos\frac{\pi t}{9}$ |
| 2 marks | Correctly determining two of the values of a, b and n |
| 1 mark | Determining the correct value of a, b or n |